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Translated by L. K.

UDC 538.4

ELECTRIC FIELD IN A MHD CHANNEL OF RECTANGULAR CROSS SECTION IN THE PRESENCE OF THE HALL EFFECT

PMM Vol. 37, №3, 1973, pp. 459-468
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(Received May 10, 1972)

The present paper deals with the spatial distribution of the electrostatic potential in a channel with two electrodes in the presence of the Hall effect. The velocity profile is inhomogeneous and corresponds to the velocity diminishing down to zero at the channel walls. The problem of determining the electric field in the channel is reduced to that of solving a boundary value problem with mixed boundary conditions for an elliptic type equation. One of the versions of the Wiener-Hopf method is used in the course of solution.

The three-dimensional distribution of the electric field in a MHD channel has been studied, because of considerable difficulties of mathematical nature encountered, only for the simplest cases of isotropically conducting media, i.e. for the cases when the walls have uniform conducting properties, or when an electrode zone is present in the channel [1 – 7]. For the anisotropic conductivity of the medium only plane problems have been studied [8, 9].

1. The case of semi-infinite electrodes. 1°. Let us consider a flow of a viscous, incompressible, anisotropically conducting medium in a MHD channel of rectangular cross section $|x| < \infty$, $|y| < b$, $|z| < 1$, in an external homogeneous magnetic field $\mathbf{H}_0$ $(0, H_0, 0)$, $H_0 =$ const. For $y = \pm b$, the channel walls are insulators, while the other two walls $(z = \pm 1)$ are insulators for $x < 0$ and perfectly conducting electrodes for $x > 0$. The velocity of the medium is

$$\mathbf{v} = (v_x, 0, 0), \qquad v_x(y, z) = \sum_{k=0}^{\infty} Z_k(z) \cos \lambda_k y$$
$$\lambda_k = k\pi / b, \quad Z_k(\pm 1) = 0, \quad Z_k(z) = Z_k(-z)$$

The distribution of the electrostatic potential $\varphi = \varphi(x, y, z)$ and the current density $\mathrm{J} = \mathrm{J}(x, y, z)$ (under the assumption that $\mathrm{Re}_m \ll 1$) are determined from the system [10]

$$\operatorname{div} \mathbf{j} = 0$$
$$\mathbf{J} = \sigma[-\nabla\varphi + \mathbf{v} \times \mathbf{H}_0] - \frac{\beta}{H_0}[\mathbf{j} \times \mathbf{H}_0] \tag{1.1}$$

(where β is the Hall's parameter), with the following boundary conditions: $j_n = 0$ at the insulators and $\varphi = \text{const}$ at the electrodes. Thus the electrostatic potential $\varphi(x, y, z)$ satisfies the boundary value problem

$$\begin{gathered}\triangle\varphi = H_0 \frac{\partial v_x}{\partial z}, \quad y^* = \frac{y}{\sqrt{1+\beta^2}}, \quad b^* = \frac{b}{\sqrt{1+\beta^2}} \\ \frac{\partial\varphi}{\partial y^*} = 0, \quad y^* = \pm b^* \\ \frac{\partial\varphi}{\partial z} = \beta\frac{\partial\varphi}{\partial x}, \quad z = \pm 1, \quad x < 0 \\ \varphi = \pm\varphi_e, \quad z = \pm 1, \quad x > 0\end{gathered} \tag{1.2}$$

where $2\varphi_e$ is the potential difference between the electrodes. A unique solution of the problem (1.2) can be obtained only if the conditions at the edge $x = 0$, $z = \pm 1$. are specified. Let [8]

$$\begin{gathered}\varphi^+ \sim |x|^{1/2+\varepsilon}, \quad x \to -0, \quad z = +1, \quad \left(\frac{\partial\varphi}{\partial z}\right)^+ \sim |x|^{-1/2-\varepsilon}, \quad x \to +0, \quad z = +1 \\ \varphi^- \sim |x|^{1/2-\varepsilon}, \quad x \to -0, \quad z = -1, \quad \left(\frac{\partial\varphi}{\partial z}\right)^- \sim |x|^{-1/2+\varepsilon}, \quad x \to +0, \quad z = -1 \\ (0 \leqslant \varepsilon < 1/2)\end{gathered} \tag{1.3}$$

We write the potential φ to be determined in the form

$$\varphi(x, y, z) = \varphi_0(x, z) + \sum_{k=1}^{\infty} \varphi_k(x, z) \cos \lambda_k y^*, \quad \lambda_k = \frac{k\pi}{b^*} \tag{1.4}$$

$$\varphi_0(x, z) = u_0(x, z) + H_0 \int_0^z Z_0(z)\, dz$$

$$\varphi_k(x, z) = u_k(x, z) - H_0 \sum_{n=1}^{\infty} \frac{a_{nk}}{\nu_n^2 + \lambda_k^2} \sin \nu_n z$$

$$a_{nk} = \frac{1}{2} \int_{-1}^{1} Z_k'(z) \sin \nu_n z\, dz \qquad \nu_n = \pi\left(n - \frac{1}{2}\right)$$

In this case the functions $u_k(x, z)$ $(k = 0, 1, 2, \ldots)$ are solutions of the boundary value problems with the mixed conditions

$$\begin{gathered}\frac{\partial^2 u_k}{\partial x^2} + \frac{\partial^2 u_k}{\partial z^2} - \lambda_k^2 u_k = 0 \\ \frac{\partial u_k}{\partial z} = \beta\frac{\partial u_k}{\partial x}, \quad z = \pm 1, \quad x < 0\end{gathered} \tag{1.5}$$

$$u_k = \mp u_k^e, \quad z = \pm 1, \quad x > 0$$

where

$$u_0^e = -\varphi_e + H_0 \int_0^1 Z_0(z)\, dz$$

$$u_k^e = H_0 \sum_{n=1}^{\infty} (-1)^n \frac{a_{nk}}{\nu_n^2 + \lambda_k^2} \quad (k \neq 0)$$

We assume that the auxilliary potential $u_k(x, z)$ satisfies the conditions at the edge of the type (1.3) as well as the conditions

$$\begin{aligned} |u_k| &< c_1 e^{\tau_+ x}, \quad \tau_+ \geqslant 0, \quad x \to -\infty \\ |u_k| &< c_2 e^{\tau_- x}, \quad \tau_- = 0, \quad x \to +\infty \end{aligned} \tag{1.6}$$

2°. Let us apply the Fourier transformation to (1.5)

$$\Phi(\alpha, z) = \Phi_+(\alpha, z) + \Phi_-(\alpha, z), \quad u_k(x, z) = \frac{1}{2\pi} \int_{i\tau-\infty}^{i\tau+\infty} \Phi(\alpha, z)\, e^{-i\alpha x} d\alpha$$

$$\Phi_-(\alpha, z) = \int_{-\infty}^{0} u_k(x, z)\, e^{i\alpha x} dx, \quad \Phi_+(\alpha, z) = \int_0^{\infty} u_k(x, z)\, e^{i\alpha x} dx \qquad (\alpha = \sigma + i\tau)$$

Here and in the following the plus subscript denotes that the given function is regular in the upper semiplane $\tau > \tau_-$ and the minus sign denotes the function regular in the lower semiplane $\tau < \tau_+$ Then we obtain the following problem for the function $\Phi(\alpha, z)$ (the prime denotes the derivative with respect to z):

$$\begin{gathered} \Phi''(\alpha, z) - \gamma^2 \Phi(\alpha, z) = 0, \quad \gamma^2 = \alpha^2 + \lambda_k^2 \\ \Phi_-'(\alpha, \pm 1) = \beta\, [u_k(-0, \pm 1) - i\alpha \Phi_-(\alpha, \pm 1)] \\ \Phi_+(\alpha, \pm 1) = \pm u_k^e / i\alpha \end{gathered} \tag{1.7}$$

Eliminating the unknown $A(\alpha)$ and $B(\alpha)$ from the relation

$$\Phi(\alpha, z) = A(\alpha) \operatorname{ch} \gamma z + B(\alpha) \operatorname{sh} \gamma z$$

and using the boundary conditions (1.7) we obtain the following initial system of functional equations

$$\psi_+(\alpha) - i\alpha\beta\Delta_-(\alpha) - K(\alpha)\Lambda_-(\alpha) = \frac{u_k^e K(\alpha)}{i\alpha} \tag{1.8}$$

$$(\gamma^2 + \alpha^2\beta^2)\Delta_-(\alpha) + i\alpha\beta\, \Psi_+(\alpha) - K(\alpha)\, \Omega_+(\alpha) = \beta u_k^e\, K(\alpha)$$

where

$$K(\alpha) = \gamma \operatorname{cth} \gamma$$

$$\psi_+(\alpha) = {}^1/_2[\Phi_+'(\alpha, 1) + \Phi_+'(\alpha, -1)] + \beta^{-2}\, [u_k(-0, 1) + u_k(-0, -1)]$$

$$\Delta_-(\alpha) = {}^1/_2[\Phi(\alpha, 1) + \Phi_-(\alpha, -1)], \quad \Lambda_-(\alpha) = {}^1/_2[\Phi_-(\alpha, 1) - \Phi_-(\alpha, -1)]$$

$$\Omega_+(\alpha) = {}^1/_2[\Phi_+'(\alpha, 1) - \Phi_+'(\alpha, -1)] + \beta^{-2}[u_k(-0, 1) - u_k(-0, -1)]$$

The system (1.8) is valid in the strip $\tau_- < \tau < \tau_+$, while $\Delta_-(\alpha)$, $\psi_+(\alpha)$, $\Lambda_-(\alpha)$ and $\Omega_+(\alpha)$ serve as the unknown functions.

3°. Let us now solve the system (1.8) using the Wiener-Hopf method. Factorization of the function $K(\alpha)$ is known [11]

$$K(\alpha) = K_k^+(\alpha) K_k^-(\alpha) \tag{1.9}$$

$$K_k^+(\alpha) = \prod_{m=1}^{\infty} \frac{\sqrt{1+\lambda_k^2/\nu_m^2} - i\alpha/\nu_m}{\sqrt{1+\lambda_k^2/\mu_m^2} - i\alpha/\mu_m}, \quad K_k^-(\alpha) = K_k^+(-\alpha), \quad \mu_n = n\pi$$

We note that for $k = 0$, the factorization of $K(\alpha)$ can be written in terms of the gamma function

$$K_0^+(\alpha) = \sqrt{\pi}\,\frac{\Gamma(1 - i\alpha/\pi)}{\Gamma(^1/_2 - i\alpha/\pi)}, \qquad K_0^-(\alpha) = K_0^+(-\alpha)$$

The functions $K_k^+(\alpha)$ are regular and have no zeros when $\operatorname{Im}\alpha > -\pi/2$, moreover $K_k^+(\alpha) \sim |\alpha|^{1/2}$ for $\alpha \to \infty$ in the upper semiplane.

Let us multiply the first equation of (1.8) by $1 / K_k^+(\alpha)$ and the second one by $1 / K_k^-(\alpha)$. Grouping the terms in the usual manner according to Wiener-Hopf method, we obtain

$$\frac{\psi_+(\alpha)}{K_k^+(\alpha)} - \beta \sum_{n=1}^{\infty} \frac{iy_{nk}K_k^+(is_{nk})}{is_{nk}+\alpha} - \frac{u_k^e K_k^-(0)}{i\alpha} = K_k^-(\alpha)\Lambda_-(\alpha) +$$

$$i\alpha\beta\frac{\Delta_-(\alpha)}{K_k^+(\alpha)} - \beta\sum_{n=1}^{\infty}\frac{iy_{nk}K_k^+(is_{nk})}{is_{nk}+\alpha} + u_k^e\,\frac{K_k^-(\alpha) - K_k^-(0)}{i\alpha}$$

$$\Omega_+(\alpha) K_k^+(\alpha) + i\alpha\beta\frac{\psi_+(\alpha)}{K_k^-(\alpha)} - \beta\sum_{n=1}^{\infty}\frac{ix_{nk}K_k^+(is_{nk})}{is_{nk}-\alpha} + \beta K_k^+(\alpha)\,u_k^e =$$

$$(\gamma^2 + \alpha^2\beta^2)\frac{\Delta_-(\alpha)}{K_k^-(\alpha)} - \beta\sum_{n=1}^{\infty}\frac{ix_{nk}K_k^+(is_{nk})}{is_{nk}-\alpha}$$

$$x_{nk} = \psi_+(is_{nk}), \quad y_{nk} = \Delta_-(-is_{nk}), \quad t_{nk}^2 = \mu_n^2 + \lambda_k^2, \quad s_{nk}^2 = \nu_n^2 + \lambda_k^2$$

The functions appearing in the left-hand sides of these relations are regular in the upper semiplane $\tau > \tau_-$ and those in the right-hand side are regular in the semiplane partially overlapping the previous one $\tau < \tau_+$ $(\tau_- < \tau_+)$. Therefore each of these functions is, when considered separately, an analytic continuation of the other function, and together they form a single entire function. According to the generalized Liouville's theorem,

$$\frac{\psi_+(\alpha)}{K_k^+(\alpha)} - \beta\sum_{n=1}^{\infty}\frac{iy_{nk}K_k^+(is_{nk})}{is_{nk}+\alpha} - \frac{u_k^e K_k^-(0)}{i\alpha} = P_m(\alpha) \tag{1.10}$$

$$(\gamma^2 + \alpha^2\beta^2)\frac{\Delta_-(\alpha)}{K_k^-(\alpha)} - \beta\sum_{n=1}^{\infty}\frac{ix_{nk}K_k^+(is_{nk})}{is_{nk}-\alpha} = P_n(\alpha)$$

The powers of the polynomials $P_m(\alpha)$ and $P_n(\alpha)$ are determined by the asymptotic behavior of each function in (1.8). Using the conditions (1.3) at the edge, we can show that $P_m(\alpha) \equiv 0$ and $P_n(\alpha) = p$ $(=\text{const})$. Let us define the constant p as follows:

$$p = -\beta\sum_{n=1}^{\infty}\frac{x_{nk}K_k^+(is_{nk})}{s_{nk}+\alpha_1}, \qquad \alpha_1 = \frac{\lambda_k}{\sqrt{1+\beta^2}}$$

because the function $\Delta_-(\alpha)$ determined in (1.10) is regular in the lower semiplane $\tau < \tau_+$. Consequently we have

$$\psi_+(\alpha) = K_k^+(\alpha)\left[\frac{u_k^e K_k^-(0)}{i\alpha} + \beta\sum_{n=1}^{\infty}\frac{iy_{nk}K_k^+(is_{nk})}{is_{nk}+\alpha}\right] \tag{1.11}$$

$$\Delta_-(\alpha) = \frac{K_k^-(\alpha)}{\gamma^2+\alpha^2\beta^2}\left[p + \beta\sum_{n=1}^{\infty}\frac{ix_{nk}K_k^+(is_{nk})}{is_{nk}-\alpha}\right]$$

Let us now set $\alpha = is_{mk}$ in the first relation of (1.11) and $\alpha = -is_{mk}$ in the second relation. Then the following system of infinite algebraic equations is obtained for x_{nk} and y_{nk}

$$\frac{x_{mk}}{K_k^+(is_{mk})} - \beta\sum_{n=1}^{\infty}\frac{y_{nk}K_k^+(is_{nk})}{s_{nk}+s_{mk}} = -\frac{u_k^e K_k^-(0)}{s_{mk}}, \qquad m = 1, 2, \ldots \tag{1.12}$$

$$(\beta^2 s_{mk}^2 + \nu_m^2)\frac{y_{mk}}{K_k^+(is_{mk})} + \beta\sum_{n=1}^{\infty} x_{nk}K_k^+(is_{nk})\left(\frac{1}{s_{nk}+s_{mk}} - \frac{1}{s_{nk}+\alpha_1}\right) = 0$$

Having obtained the functions $\psi_+(\alpha)$ and $\Delta_-(\alpha)$ we can find the transform $\Phi(\alpha, z)$

$$\Phi(\alpha, z) = -u_k^e\left[\frac{K_k^-(\alpha)S_k(\alpha)}{\gamma^2+\alpha^2\beta^2}\left(\frac{\operatorname{ch}\gamma z}{\operatorname{ch}\gamma} - i\alpha\beta\frac{\operatorname{sh}\gamma z}{\gamma\operatorname{ch}\gamma}\right) + K_k^+(\alpha)R_k(\alpha)\frac{\operatorname{sh}\gamma z}{\gamma\operatorname{ch}\gamma}\right]$$

$$R_k(\alpha) = -\frac{K_k^-(0)}{i\alpha} + \beta\sum_{n=1}^{\infty}\frac{iy_{hk}K_k^+(is_{nk})}{is_{nk}+\alpha}$$

$$S_k(\alpha) = \beta\sum_{n=1}^{\infty} x_{nk}K_k^+(is_{nk})\left(\frac{1}{s_{nk}+i\alpha} - \frac{1}{s_{nk}+\alpha_1}\right)$$

We perform the inverse Fourier transformation. Then the required distribution of electrostatic potential (1.4) has the form

$$\begin{aligned}\varphi_0(x, z) = H_0\int_0^z Z_0(z)\,dz - \frac{u_0^e}{1+\beta^2}\Bigg[\beta\sum_{n=1}^{\infty}\frac{x_{n0}K_0^+(i\nu_n)}{\nu_n^2} + \\ \sum_{n=1}^{\infty}(-1)^n\frac{S_0(i\mu_n)e^{\mu_n x}}{\mu_n K_0^+(i\mu_n)}(\cos\mu_n z + \beta\sin\mu_n z) + \\ \sum_{n=1}^{\infty}(-1)^n\frac{x_{n0}}{\nu_n}e^{\nu_n x}(\beta\cos\nu_n z - \sin\nu_n z)\Bigg]\end{aligned} \tag{1.13}$$

$$\begin{aligned}\varphi_k(x, z) = -H_0\sum_{n=1}^{\infty}\frac{a_{nk}}{\nu_n^2+\lambda_k^2} - \frac{u_k^e}{1+\beta^2}\Bigg[\frac{\beta S_k(i\alpha_1)e^{\alpha_1 x}}{2\operatorname{sh}\beta\alpha_1 K_k^+(i\alpha_1)}(\operatorname{ch}\beta\alpha_1 z + \operatorname{sh}\beta\alpha_1 z) + \\ \sum_{n=1}^{\infty}\frac{(-1)^n\mu_n S_k(it_{nk})e^{t_{nk}x}}{t_{nk}K_k^+(it_{nk})(t_{nk}^2-\alpha_1^2)}(\mu_n\cos\mu_n z + \beta t_{nk}\sin\mu_n z) + \\ \sum_{n=1}^{\infty}(-1)^n\frac{\nu_n x_{nk}e^{s_{nk}x}}{s_{nk}(s_{nk}^2-\alpha_1^2)}(\beta s_{nk}\cos\nu_n z - \nu_n\sin\nu_n z)\Bigg]\end{aligned}$$

for the region $x < 0$ and

$$\varphi_0(x, z) = H_0\int_0^z Z_0(z)\,dz -$$

$$u_0^e\left[z - \sum_{n=1}^{\infty}(-1)^n y_{n0}e^{-\nu_n x}\cos\nu_n z - \sum_{n=1}^{\infty}(-1)^n\frac{R_0(-i\mu_n)}{K_0^+(i\mu_n)}e^{-\mu_n x}\sin\mu_n z\right]$$

$$\varphi_k(x, z) = -H_0\sum_{n=1}^{\infty}\frac{a_{nk}}{\nu_n^2-\lambda_k^2}\sin\nu_n z - u_k^e\left[\frac{\operatorname{sh}\lambda_k z}{\operatorname{sh}\lambda_k} - \right. \qquad (1.14)$$

$$\left.\sum_{n=1}^{\infty}(-1)^n\frac{y_{nk}\nu_n}{s_{nk}}e^{-s_{nk}x}\cos\nu_n z - \sum_{n=1}^{\infty}(-1)^n\frac{\mu_n R_k(-it_{nk})\,e^{-t_{nk}x}}{t_{nk}K_k^+(it_{nk})}\sin\mu_n z\right]$$

for the region $x > 0$.

4°. Next we consider the infinite systems (1.12). We replace x_{nk} and y_{nk} by introducing new unknown $x'_{nk} = -\dfrac{x_{nk}}{u_k^e K_k^-(0)}$, $\quad y'_{nk} = -\dfrac{y_{nk}(s_{nk}+\alpha_1)}{u_k^e K_k^-(0)}$

For the latter we obtain from (1.10) a system which, when solved for each unknown, yields two infinite systems of equations of the following form:

$$x_m = \sum_{n=1}^{\infty}c_{mn}x_n + b_m \qquad (1.15)$$

where

$$c_{mn} = \frac{\beta^2}{1+\beta^2}\frac{K_k^+(is_{mk})\,K_+^2(is_{nk})}{(s_{nk}+\alpha_1)(s_{nk}+s_{mk})}\sum_{t=1}^{\infty}\frac{K_k^+(is_{tk})}{(s_{tk}+\alpha_1)(s_{tk}+s_{nk})}$$

The systems (1.15) are completely regular for any $0 \leqslant \beta < \infty$. This follows from the estimate

$$\sum_{n=1}^{\infty}c_{mn} \leqslant \frac{\beta^2}{1+\beta^2}K_k^+(is_{mk})\sum_{n=1}^{\infty}\frac{K_+^2(is_{nk})}{s_{nk}(s_{nk}+s_{mk})}\sum_{t=1}^{\infty}\frac{K_k^+(is_{tk})}{s_{tk}(s_{tk}+s_{nk})} = \frac{\beta^2}{1+\beta^2} < 1$$

The above estimate was proved using the equation

$$\sum_{n=1}^{\infty}\frac{K_k^+(is_{nk})}{s_{nk}(s_{nk}+s_{mk})} = \frac{1}{K_k^+(is_{mk})}$$

obtained while computing the following contour integral [11] with the aid of the theory of residues:

$$f_-(\alpha) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f_-(\zeta)\,d\zeta}{\zeta-\alpha}, \quad \tau < 0$$

where

$$f_-(\alpha) = 1\,/\,K_k^-(\alpha), \quad \alpha = -is_{nk}$$

The free terms b_m are bounded within a set, consequently the solution of (1.15) can be obtained by the method of reduction or the method of consecutive approximations [12]. Moreover, it can be shown that the inequalities $x'_{n,k-1} > x'_{n,k}$ and $y'_{n,k-1} > y'_{n,k}$. are valid.

2. The case of finite electrodes. We consider, as before, a rectangular channel $|x| < \infty$, $|y| < b$, $|z| < 1$ the walls of which are nonconducting everywhere except for two symmetrically placed electrodes $z = \pm 1$, $|x| < a$. The boundary value problem is written in the form

$$\frac{\partial^2 u_k}{\partial x^2}+\frac{\partial^2 u_k}{\partial z^2}-\lambda_k^2 u_k=0$$
$$\frac{\partial u_k}{\partial z}=\beta\frac{\partial u_k}{\partial x},\quad z=\pm 1,\quad |x|>a \tag{2.1}$$
$$u_k=\mp 1,\quad z=\pm 1, |x|<a$$

Applying the Fourier transformation in x, we obtain the following system of functional equations:

$$(\alpha^2\beta^2+\gamma^2)\Delta_+(\alpha)e^{i\alpha a}+i\alpha\beta R_0(\alpha)-K(\alpha)S_0(\alpha)+$$
$$(\alpha^2\beta^2+\gamma^2)\Delta_-(\alpha)e^{-i\alpha a}=-2i\beta K(\alpha)\sin\alpha a \tag{2.2}$$
$$(\alpha^2\beta^2+\gamma^2)\ \Delta_+(\alpha)e^{i\alpha a}+\ i\alpha\beta S_0\ (\alpha)-\gamma^2 R_0(\alpha)/K\ (\alpha)+$$
$$(\alpha^2\beta^2+\gamma^2)\Lambda_-(\alpha)\ e^{-i\alpha a}=\gamma^2\alpha^{-1}\sin a$$

for the unknown functions

$$\Delta_\pm(\alpha)={}^1/_2[\Phi_\pm(\alpha,1)+\Phi_\pm(\alpha,-1)]$$
$$\Lambda_\pm(\alpha)={}^1/_2[\Phi_\pm(\alpha,1)-\Phi_\pm(\alpha,-1)]$$
$$R_0(\alpha)={}^1/_2[\Phi_0'(\alpha,1)+\Phi_0'(\alpha,-1)]-{}^1/_2\beta e^{i\alpha a}[u_k(a+0,1)+$$
$$u_k(a+0,-1)+{}^1/_2\beta e^{-i\alpha a}[u_k(-a-0,+1)+u_k(-a-0,-1)],$$
$$S_0(\alpha)={}^1/_2[\Phi'_0(\alpha,1)-\Phi_0'(\alpha,-1)-{}^1/_2\beta e^{-i\alpha a}[u_k(a+0,1)-$$
$$u_k(a+0,-1)+{}^1/_2\beta e^{-i\alpha a}[u_k(-a-0,1)-u_k(-a-0,-1)]$$

where

$$\Phi_-(\alpha,z)=\int_{-\infty}^{-a}u_k(x,z)e^{i\alpha(x+a)}dx,\quad \Phi_0(\alpha,z)=\int_{-a}^{a}u_k(x,z)e^{i\alpha x}dx$$
$$\Phi_+(\alpha,z)=\int_{a}^{\infty}u_k(x,z)e^{i\alpha(x-a)}dx$$

Equations (2.2) are valid in the strip $\tau_-<\tau<\tau_+$, the functions $\Delta_+(\alpha)$ and $\Lambda_+(\alpha)$ are regular for $\tau>\tau_-$, $\Delta_-(\alpha)$ and $\Lambda_-(\alpha)$ are regular for $\tau<\tau_+$, while $S_0(\alpha)$ and $R_0(\alpha)$ are entire functions. The function $K(\alpha)=K_+(\alpha)K_-(\alpha)$ is defined by relation (1.9). Equations (2.2) are solved using a method given in [13] generalized to embrace the case of systems of functional equations. The computations are cumbersome and the solution is therefore not given here. We shall just mention that it agrees to within the terms of the order $O(e^{-2\pi a})$ with the approximate solution which can be obtained in the following manner.

The solution (1.13),(1.14) for the entry zone can easily be transformed into a solution corresponding to the exit zone. To do this, it is sufficient to change the signs of the Hall parameter and of the variable x in the solution indicated. We therefore have the following approximate solution of the problem for the channel with finite electrodes of length $2a$:

$$\varphi_0(x,z)=H_0\int_0^z Z_0(z)\,dz-\frac{u_0^e}{1+\beta^2}\Big[\pm\beta\sum_{n=1}^{\infty}\frac{x_{n0}K_0^+(i\nu_n)}{\nu_n^2}+$$
$$\sum_{n=1}^{\infty}\frac{(-1)^n S_0(i\mu_n)e^{\mu_n(a-|x|)}}{\mu_n K_0^+(i\mu_n)}(\beta\sin\mu_n z\pm\cos\mu_n z)-$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n x_{n0}}{\nu_n} e^{\nu_n(a-|x|)} (\sin \nu_n z \mp \beta \cos \nu_n z)]$$

$$\varphi_k(x, z) = -\sum_{n=1}^{\infty} \frac{a_{nk} \sin \nu_n z}{\nu_n^2 + \lambda_k^2} - \frac{u_k^e}{1+\beta^2} \Big[\frac{\beta S_k(i\alpha_1) e^{\alpha_1(a-|x|)}}{2 \operatorname{sh} \beta\alpha_1 K_k^+(i\alpha_1)} (\operatorname{sh} \beta\alpha_1 z \pm$$

$$\operatorname{ch} \beta\alpha_1 z) + \sum_{n=1}^{\infty} \frac{(-1)^n \mu_n S_k(it_{nk}) e^{t_{nk}(a-|x|)}}{t_{nk} K_k^+(it_{nk})(t_{nk}^2 - \alpha_1^2)} (\beta t_{nk} \sin \mu_n z \pm \cos \mu_n z) - \quad (2.3)$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{\nu_n x_{nk} e^{s_{nk}(a-|x|)}}{s_{nk}(s_{nk}^2 - \alpha_1^2)} (\nu_n \sin \nu_n z \mp \beta s_{nk} \cos \nu_n z)$$

for the region $|x| > a$ (here the upper sign corresponds to the region $x < -a$) and

$$\varphi_0(x, z) = H_0 \int_0^z Z_0(z)\, dz - u_0^e \Big[z + \sum_{n=1}^{\infty} (-1)^n 2y_{n0} e^{-\nu_n a} \operatorname{sh} \nu_n x \cos \nu_n z -$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2R_0(-i\mu_n)}{K_0^+(i\mu_n)} e^{-\mu_n a} \sin \mu_n z \operatorname{ch} \mu_n x \Big] \quad (2.4)$$

$$\varphi_k(x, z) = -H_0 \sum_{n=1}^{\infty} \frac{a_{nk}' \sin \nu_n z}{\nu_n^2 + \lambda_k^2} - u_k^e \Big[\frac{\operatorname{sh} \lambda_k z}{\operatorname{sh} \lambda_k} +$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2y_{nk}\nu_n}{s_{nk}} e^{-s_{nk} a} \operatorname{sh} s_{nk} x \cos \nu_n z -$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2\mu_n R_k(-it_{nk})}{t_{nk} K_k^+(it_{nk})} e^{-t_{nk} a} \operatorname{ch} t_{nk} x \sin \mu_n z \Big]$$

for the region $|x| < a$

3. Effect of the anisotropy of the conductivity of the medium on the integral characteristics of the three-dimensional channel.

In computing the integral characteristics of the channel we shall limit ourselves, for definiteness, to considering the following velocity profile

$$\mathbf{v}(v_x, 0, 0), \quad v_x(y, z) = \frac{3}{2} \delta u_0 \frac{\operatorname{ch} \delta - \operatorname{ch} \delta y / b}{\delta \operatorname{ch} \delta - \operatorname{sh} \delta} (1 - z^2)$$

Here $\delta > 0$ is the profile-leading parameter and u_0 is the velocity averaged over a cross section. We also assume that the MHD channel works in the generating mode and the length of the electrode zone is not less than the distance between the electrodes. Thus the distribution of electrostatic potential is defined by (2.3) and (2.4) with the accuracy of up to the terms of the order $O(e^{-2\pi a})$. The integral characteristics sought are: the potential φ_e at the electrodes, total current and power through the external load R, the Joule dissipation and the efficiency (efficiency factor) of the generator channel. The potential at the electrodes can be expressed in terms of the load coefficient k in the following manner: $\varphi_e = ku_0H_0$ and the total current I flowing through the electrodes into the external network is computed by the formula

$$I = \int_{-a}^{a}\int_{-b}^{b} j_z(x, y, 1)\,dx\,dy = \int_{-a}^{a}\int_{-b}^{b} j_z(x, y, -1)\,dx\,dy =$$
$$\frac{4b\sigma u_0 H_0}{1+\beta^2}(1-k)\left(a + 0.441 + \beta \sum_{n=1} \frac{y_{n0}}{\nu_n}\right)$$

where the current density $j_z(x, y, z)$ is determined from (1.1). Then the expression for the power N and the Joule dissipation Q in the channel can be written in the form

$$N = 2\varphi_e I = \frac{8b\sigma H_0^2 u_0^2}{1+\beta^2}\, k(1-k)\left(a + 0.441 + \beta \sum_{n=1}^{\infty} \frac{y_{n0}}{\nu_n}\right)$$

$$Q = \int_D \sigma^{-1} j^2\, dD = -N - \int_D [\mathbf{j} \times \mathbf{H}]\, \mathbf{v}\, dD = Q_* + \Delta Q$$

$$\frac{Q_*}{4\sigma^* b H_0^2 u_0^2} = -2\frac{k^2}{\omega} + 2(1-k)\left[a - 0.054 + \sum_{n=1}^{\infty} \frac{x_{n0}}{\nu_n^4}\left(1 - e^{-\nu_n(L-a)}\right) - \beta \sum_{n=1}^{\infty} \frac{y_{n0}}{\nu_n^3} P_2(\nu_n)\right]$$

$$\sigma^* = \frac{\sigma}{1+\beta^2}, \qquad \omega = 2b\sigma^* R, \qquad P_2(\nu_n) = 0.208\nu_n^2 - 1.323\nu_n - 3$$

$$\Delta Q = \frac{4\sigma b}{1+\beta^2}\left\{\frac{6L}{5\delta} H_0^2 u_0^2 + o(\delta^{-2}) + H_0^2 \sum_{k=1}^{\infty} \frac{a_k^2}{\lambda_k^5}(\lambda_k \operatorname{cth} \lambda_k + \operatorname{th} \lambda_k - 2\lambda_k) - \right.$$
$$H_0^2 L \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2}\left[\frac{1}{3} + \frac{\operatorname{th}\lambda_k - \lambda_k}{\lambda_k^5}\right] + H_0^2 \sum_{k=1}^{\infty} \frac{a_k^2}{\lambda_k^2}\left(\frac{\operatorname{sh}\lambda_k}{\lambda_k \operatorname{ch}\lambda_k} - 1\right) \times$$
$$\left[\frac{i}{\lambda_k^2}\frac{dK_k^-}{d\alpha}(0)\, K_k^-(0) + \frac{K_k^+(i\lambda_k) - K_k^-(i\lambda_k)}{2\lambda_k^3} K_k^-(0) - \sum_{n=1}^{\infty} \frac{x_{nk}}{s_{nk}^2 \nu_n^2}\left(1 - e^{-(L-a)s_{nk}}\right) - \right.$$
$$\left.\left.\beta \sum_{n=1}^{\infty} \frac{y_{nk}}{s_{nk}\nu_n^2} + \beta \sum_{n=1}^{\infty} \frac{y_{nk} K_k^+(is_{nk})}{2\lambda_k^2}\left(\frac{K_k^+(i\lambda_k)}{s_{nk} + \lambda_k} + \frac{K_k^-(i\lambda_k)}{s_{nk} - \lambda_k} - \frac{2K_k^-(0)}{s_{nk}}\right)\right]\right\}$$

where Q_* denotes the Joule losses in a plane channel $(\delta \to \infty)$ due to the longitudinal edge effect only, $2L$ is the generator channel length $(L > a)$, and ΔQ is the dissipation increment due to both,the longitudinal and the transverse flows.

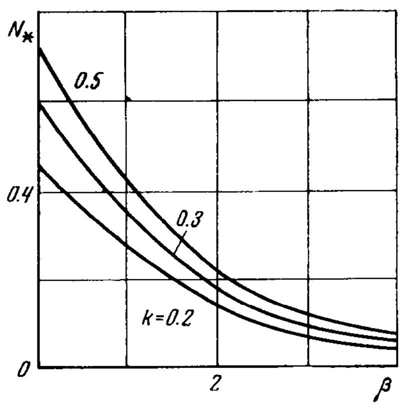

Fig. 1

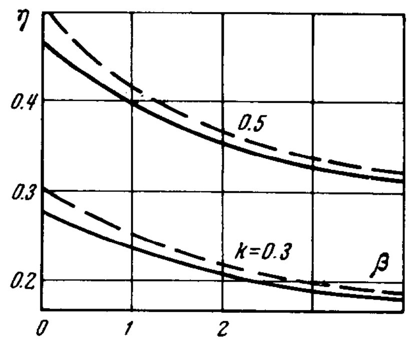

Fig. 2

The generator efficiency (efficiency factor) is found from the formula

$$\eta = N / (N + Q)$$

Figures 1 and 2 depict the dependence of the power $(N_* = N / (48\sigma H_0^2 U_0^2))$ and the efficiency factor of the channel on the Hall parameter, for certain values of the load coefficient k, with the solid lines corresponding to the three-dimensional channel of length $L = 4a$ and $\delta = 100$, and the broken lines corresponding to a plane channel. Table 1 gives a solution of the infinite system of equations (1.15) for $(k = 0)$.

Table 1

	x_{n0}					y_{n0}			
n	β				n	β			
	0.1	1	3	5		0.1	1	3	5
1	1.005	1.367	2.131	2.359	1	0.063	0.467	0.469	0.448
2	0.504	0.831	1.528	1.737	2	0.052	0.400	0.413	0.344
3	0.379	0.683	1.338	1.534	3	0.046	0.362	0.381	0.302
4	0.316	0.605	1.227	1.414	4	0.042	0.336	0.357	0.277
5	0.277	0.553	1.150	1.329	5	0.039	0.316	0.339	0.260
6	0.249	0.515	1.091	1.264	6	0.037	0.300	0.325	0.246
7	0.229	0.485	1.043	1.211	7	0.035	0.287	0.313	0.236
8	0.199	0.461	1.003	1.166	8	0.034	0.276	0.302	0.227
9	0.196	0.440	0.968	1.127	9	0.032	0.267	0.293	0.219
10	0.188	0.423	0.938	1.093	10	0.031	0.258	0.285	0.212

Thus the presence of anisotropic conductivity in the medium affects the integral characteristics of the MHD channel unfavorably. As in the case of a plane channel [10], the edge losses near solid electrodes increase appreciably with the increase in the value of the Hall parameter β.

In conclusion the author thanks L. P. Chegirin for kindly solving the infinite system of algebraic equations (1.15) on a computer, and A. A. Kaspar'iants for continuous interest in this work.

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Translated by L. K.

UDC 533.6.011

ON CONSTRUCTING THE CONTOUR OF MINIMUM WAVE DRAG IN AN INHOMOGENEOUS SUPERSONIC FLOW

PMM Vol. 37, №3, 1973, pp. 469-487
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(Received December 18, 1972)

A variational problem is considered of constructing the generatrix of a plane or axisymmetric body guaranteeing the minimum wave drag in an inhomogeneous (nonisentropic and nonisoenergetic) supersonic flow of an ideal gas (inviscid and non-heat-conducting) in the case when the domain of determinacy of the unknown contour contains a zone of sharp variation in the values of parameters which are retained (in the absence of jumps) along the streamline, the parameters being the entropy and the stagnation enthalpy. In the limit the zone degenerates to a tangential discontinuity. The investigation is limited to the configurations (e. g. nozzles or the stern parts of the bodies) for which no shock waves (this includes the bow shock) exist in the region under investigation. It is established that the solution [1, 2] obtained earlier for inhomogeneous flows and yielding a smooth optimal countour (without internal corner points) cannot be realized in such cases and must be replaced by a solution in which the generatrix of the optimal body contains at least one internal corner point. Since the method of passing to the control contour utilized in [1, 2] cannot be applied to the study of such configurations, the necessary extremal conditions determining the form of the optimal generatrix must be obtained using the general method of Lagrange muliplier s in the form developed in [3 – 5]. The conditions of optimal-